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A numerical solution of the Boussinesq equation using the Adomian method

A. G. Bratsos [1], **D. P. Papadopoulos** [2] **and Ch. Skokos** [3]

[1,2]Department of Mathematics,
Technological Educational Institution (T.E.I.) of Athens,
GR-122 10 Egaleo, Athens, Greece.
[3]Research Center for Astronomy and Applied Mathematics
Soranou Efesiou 4, GR-11527, Athens, Greece

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Abstract: The Adomian decomposition method is applied to a known from the bibliography finite-difference scheme concerning the numerical solution of the Boussinesq equation. The method is examined numerically for both the single and the double-soliton waves.

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1 Introduction

In the past few years interest has increased in the solution of partial differential equations governing non-linear waves in dispersive media. As a result several texts and numerous research papers have been devoted to the subject. In parallel with the mathematical treatment a considerable literature has grown dealing with the numerical solution of such problems. Among them a great interest has been developed for equations, which possess special solutions in the form of pulses, which retain their shapes and velocities after interaction amongst themselves. Such solutions are called *solitons*. Solitons are of great importance in many physical areas, as for example, in dislocation theory of crystals, plasma and fluid dynamics, magnetohydrodynamics, laser and fiber optics etc., as well as in the study of the water waves. The development of analytical solutions of soliton type equations has been with us for many years (see for example Ablowitz and Segur [1] who implemented the inverse scattering transform method, Hirota [9] who constructed the N soliton solutions using the bilinear form, as well as Whitman [17] etc.). A part of these equations is going to be examined at the present paper.

The archetypal equation introduced by Korteweg & de Vries (KdV) [11], which describes long gravity waves moving over stationary water is written as

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0\,, \quad x \in \Re \tag{1}$$

[1]Corresponding author. E-mail: bratsos@teiath.gr

Once the general method of solution of the KdV equation was obtained (see Gardner [8]), many other equations and mathematical approaches followed. In the particular field of water waves, two families of evolution equations occur: one is the KdV family of equations and the other is based on the nonlinear Schrödinger (NLS) equation

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+u|u|^2=0.$$

The Boussinesq (BS) nonlinear equation, which belongs to the KdV family of equations, describes shallow water waves propagating in both directions, is given by

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+q\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2\left(u^2\right)}{\partial x^2}\ ;\quad L_0<x<L_1,\ t>0. \tag{2}$$

where $u=u\left(x,t\right)$ is a sufficiently often differentiable function and $|q|=1$ is a real parameter. Taking $q=-1$ gives the *Good Boussinesq* or *well-posed* equation (GB), while taking $q=1$ gives the *Bad Boussinesq* or *ill-posed* equation (BB). Besides to Hirota [9] bilinear formalism, Nimmo and Freeman [13] introduced an alternative formulation of the N-soliton solutions in terms of some function of the Wronskian determinant of N functions, Kaptsov [10] implemented Hirota's method to construct a new set of exact solutions of BS equation, while recently, among others, numerical solutions of the BS equation have been given by Bratsos [3] using the method of lines and Wazwaz [15] using the Adomian decomposition method.

The initial displacements associated with BS equation will be assumed to have the form,

$$u\left(x,0\right)=g(x)\ \text{ and }\ \frac{\partial u\left(x,0\right)}{\partial t}=\hat{g}(x)\ ;\ L_0\le x\le L_1. \tag{3}$$

1.1 The single-soliton solution

Following Manoranjan et *al* [12] the theoretical solution of BS equation is given by

$$u(x,t)=q_1\left\{A\operatorname{sech}^2\left[\sqrt{\frac{A}{6}}\left(x-ct+x_0\right)\right]+\left(b-\frac{q_1}{2}\right)\right\}. \tag{4}$$

where A is the amplitude of the pulse, b is an arbitrary parameter, x_0 is the initial position of the pulse and $c=\pm\left[2q_1\left(b+A/3\right)\right]^{1/2}$, where $q_1=1$ for the BB and $q_1=-1$ for the GB equation.

1.2 The double-soliton solution

Similarly following Hirota [9] and Manoranjan et *al* [12], who obtained a double-soliton solution for GB, the double-soliton solution for both GB and BB can be written as,

$$u\left(x,t\right)=6q_1\frac{\partial^2}{\partial x^2}\left[\log_e f\left(x,t\right)\right], \tag{5}$$

where $f\left(x,t\right)=1+\exp\left(\eta_1\right)+\exp\left(\eta_2\right)+\alpha\exp\left(\eta_1+\eta_2\right)$ with $\eta_i=P_i\left[x-\epsilon_i\left(1+q_1P_i^2\right)^{1/2}t-q_1x_i^0\right]$ for $i=1,2$ and $\epsilon_i=+1$ or -1 showing the direction in which the two solitons are traveling,

$$\alpha=\frac{\left(\epsilon_1 v_1-\epsilon_2 v_2\right)^2+3q_1\left(P_1-P_2\right)^2}{\left(\epsilon_1 v_1-\epsilon_2 v_2\right)^2+3q_1\left(P_1+P_2\right)^2}\ .$$

in which $v_i=\left(1+q_1P_i^2\right)^{1/2}$ and $P_i^2=\frac{2}{3}A_i$; $i=1,2$, where A_i is the amplitude and x_i^0 the initial position of the i-th soliton (see also Bratsos [4]).

2 The Adomian method

Consider Eq. (2) written in an operator form as

$$\mathrm{L}\,u = \frac{\partial^2 u}{\partial x^2} + q\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 \left(u^2\right)}{\partial x^2}\ ; \quad L_0 < x < L_1,\ t > 0, \tag{6}$$

where

$$\mathrm{L} = \frac{\partial^2}{\partial t^2}, \tag{7}$$

is a twice integrable differential operator with

$$\mathrm{L}^{-1}(.) = \int_0^t \int_0^t (.)\, dt\, dt. \tag{8}$$

Following the Adomian decomposition method (see Adomian [2]) the unknown solution function u is assumed to be given by a series of the form

$$u\,(x,t) = \sum_{n=0}^{\infty} u_n\,(x,t)\,, \tag{9}$$

where the components $u_n\,(x,t)$ are going to be determined recurrently, while the nonlinear term $F\,(u) = \left(u^2\right)_{xx}$ in Eq. (2) is decomposed into an infinite series of polynomials of the form

$$F\,(u) = \sum_{n=0}^{\infty} A_n, \tag{10}$$

with A_n the so-called Adomian polynomials of $u_0,\ u_1,\ \ldots,\ u_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{\infty}\lambda^i\,u_i\right)\right]_{\lambda=0} \quad \text{for } n = 0, 1, 2, \ldots \tag{11}$$

and constructed for all classes of nonlinearity according to algorithms given either by Adomian [2] or alternatively by Wazwaz [14], [16].

3 Numerical experiments

The Adomian decomposition method for solving the BS equation was tested numerically to the problems proposed by Bratsos [3], [4] with boundary lines $L_0 = -80$ and $L_1 = 100$, initial conditions defined by Eq. (3) and theoretical solutions given by Eq. (4) for the single and Eq. (5) for the double-soliton waves.

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